Quotient of Operators

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Overview

The following notions are introduced for a pair of bounded linear operators on a Hilbert space.

- Left quotient of operators
- Right quotient of operators

We discuss few results on left and right quotient of operators.

Notations

- H, a Hilbert space (not necessarily separable) over the field $\mathbb{K}$ of real or complex scalars.
- $B(H)$, the Banach space of all bounded linear operators on H
- $B_c(H)$, operators in $B(H)$ with closed range
- $D(T)$, the domain of an operator T
- $R(T)$, the range of T
- $N(T)$, the null space of T

Introduction

Let A and B be bounded linear operators on a Hilbert space H with the **kernel condition**

$$N(A) \subseteq N(B).$$

The **quotient operator** $[B/A]$ of A and B is defined as the mapping $Ax \mapsto Bx, x \in H$.

If we write $G(A, B)$ for the set $\{(Ax, Bx) : x \in H\}$ in the product Hilbert space $H \times H$, then $G(A, B)$ is a graph and we can define $[B/A]$ as the operator corresponding to this graph.

Note that the quotient operator $[B/A]$ is not necessarily bounded.

Introduction

A quotient (of bounded operators) so defined appeared for the first time in the work of Dixmier [1] by the name "operateur J uniforme" and investigated by Kaufman under the name "semiclosed operators."

Ever since the publication of Kaufman's seminal paper [2], there has been a continuing interest in the representation of closed and semiclosed linear operators on H as quotients of bounded operators.

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[1] J. Dixmier, "*Etude sur les varietes et les operateurs de Julia avec quelques applications* 2," (French), *Bull. Soc. Math. France*, 1949 (77), 11–101.

[2] W.E. Kaufman, "*Representing a closed operator as a quotient of continuous operators,*" Proc. Amer. Math. Soc. 72 (1978), 531-534.

Applications

Applications include investigations of topologies on the sets of unbounded linear Hilbert space operators, selfadjoint extensions of positive quotients, study of weak adjoints of operator quotients, algebraic properties of quotients, and various other topics.

Theorem 1 (Douglas, 1966).

Let A and B be bounded operators on a Hilbert space H. The following statements are equivalent:

1. $R(A) \subseteq R(B)$.
2. *There exists* $M > 0$ *such that* $\|A^*x\| \leq M \, \|B^*x\|$ *for all* $x \in H$. *That is, there exists* $M > 0$ *such that* $AA^* \leq M^2 \, BB^*$.
3. *There exists a bounded operator* C *on* H *so that* $A = BC$.

Uniqueness of the operator C

Moreover, if one of the conditions holds, then there exists a unique operator C such that

(a) $\|C\|^2 = \inf\{\mu : AA^* \leq \mu\, BB^*\}$;

(b) $N(C) = N(A)$; and

(c) $R(C) \subseteq \overline{R(B^*)}$.

We shall call this uniquely determined C, the **Douglas solution** of the operator equation $A = BX$.

The operator C is uniquely determined by the condition (c) and satisfies the two conditions (a) and (b).

In general, the operator C may not be unique.

Left quotient of bounded operators

Definition 2 ([7]).

Let $A \in B(H)$ *and* $B \in B(H)$ *such that*

$$R(A) \subseteq R(B) \qquad \textbf{(the range inclusion holds)}.$$

The unique bounded operator C *as a Douglas solution of* $A = BX$ *is called the* **left quotient** *of* A *by* B *and it is denoted by* $[A \backslash B]$*. Here the symbol "* $\backslash$*" is "backslash."*

1. $B[A\backslash B]x = Ax$, for all $x \in H$.
2. $[A\backslash I] = A$.
3. The left quotient operator is always bounded.

Definition 3.

If $A \in B_c(H)$, then $A^\dagger$ is the unique linear operator in $B_c(H)$ satisfying

1. $AA^\dagger A = A$;
2. $A^\dagger AA^\dagger = A^\dagger$;
3. $AA^\dagger = (AA^\dagger)^*$;
4. $A^\dagger A = (A^\dagger A)^*$.

The operator $A^\dagger$ is called the Moore-Penrose inverse of A.

Left quotient via Moore-Penrose inverse

Theorem 4 ([7]).

Let $A \in B(H)$ and $B \in B_c(H)$ such that $R(A) \subseteq R(B)$. Then $[A\backslash B] = B^\dagger A$. In particular, if B is invertible, then $[A\backslash B] = B^{-1}A$.

Corollary 5 ([7]).

Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times p}$ such that $R(A) \subseteq R(B)$. Then $[A\backslash B] = B^\dagger A \in \mathbb{C}^{p\times m}$.

Corollary 6 ([7]).

Let $A \in B(H)$ and $B \in B(H)$ have the same range. Then $[A\backslash B]$ is an invertible operator $N(A)^\perp$ to $N(B)^\perp$ with

$$[A\backslash B]^{-1} = [B\backslash A] : N(B)^\perp \to N(A)^\perp.$$

In particular, if B is invertible, then $[A\backslash B] = B^{-1}A$.

Right quotient of bounded operators

Definition 7.

Let A and B be bounded operators on a Hilbert space H such that

$$N(A) \subseteq N(B) \qquad \textbf{(the kernel inclusion holds)}.$$

The **right quotient** $[B/A]$ *of A by B is defined as the mapping*

$$Ax \mapsto Bx, \quad x \in H.$$

(a) The domain, range and nullspaces of $[B/A]$ are respectively $R(A), R(B)$ and $A(N(B))$.

(b) For all $x \in H$, we have $[B/A]Ax = Bx$. So, the right quotient $[B/A]$ is the unique solution of the equation $B = XA$.

(c) Note that for $A \in B(H)$, $[A/I] = A$.

We say that $[D/C]$ is an **extension** of $[B/A]$ if $G(A, B)$ is a subspace of $G(C, D)$. It is denoted by $[B/A] \subseteq [D/C]$.

Proposition 8.

$[B/A] \subseteq [D/C]$ if and only if there eixsts $X \in B(H)$ such that

$$A = CX \quad \text{and} \quad B = DX.$$

The quotient operator $[B/A]$ of A and B is not necessarily bounded.

Theorem 9.

Let $A \in B(H)$ and $B \in B(H)$ with $N(A) \subseteq N(B)$. Then $[B/A]$ is bounded if and only if $R(B^*) \subseteq R(A^*)$.

Right Quotient via Moore-Penrose inverse

Theorem 10.

Let $A \in B_c(H)$ and $B \in B(H)$ such that $N(A) \subseteq N(B)$. Then $[B/A] = BA^\dagger$. In particular, if A is invertible, then $[B\backslash A] = BA^{-1}$.

Corollary 11.

Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times n}$ such that $N(A) \subseteq N(B)$. Then $[B/A] = BA^\dagger \in \mathbb{C}^{p\times m}$.

Corollary 12.

Let $A \in B(H)$ and $B \in B(H)$ have the same kernel. Then $[B/A]$ is an invertible operator $R(A)$ to $R(B)$ with $[B/A]^{-1} = [A/B]$.
In particular, if $R(A)$ is closed, then $[B/A]^{-1}$ is bounded.

Right Quotient via Moore-Penrose inverse

Theorem 13.

Let $A \in B(H)$ and $B \in B(H)$ such that $N(A) \subseteq N(B)$. Then $[B \backslash A]$ is closed if and only if $R(A^*) + R(B^*)$ is closed in H.

Adjoint of Left Quotient Operator

Theorem 14.

Let $A \in B(H)$ and $B \in B(H)$ such that $R(A) \subseteq R(B)$. Then

$$[B\backslash A]^* = [A^*\backslash B^*].$$

Also, if $R(B^*)$ is closed, then

$$[B\backslash A]^* = A^*(B^*)^\dagger.$$

Adjoint of Right Quotient Operator

Theorem 15.

Let $A \in B(H)$ and $B \in B(H)$ such that $N(A) \subseteq N(B)$.

1. *The adjoint of the right quotient $[B/A]$ exists and is closed from $\overline{R(B)}$ to $\overline{R(A)}$.*
2. *If $R(A^*) + R(B^*)$ is closed in H, then $[B/A]^*$ is a closed densely defined operator from H to $\overline{R(A)}$.*
3. *If $R(A^*)$ is closed in H, then*

$$[B/A]^* = [A^* \backslash B^*] = (A^*)^\dagger B^*$$

and

$$[B/A]^{**} = [B/A].$$

Semiclosed Operator

Definition 16 ([5]).

A subspace M (need not be closed) of a Hilbert space $(H, \langle ., . \rangle)$ is called semiclosed if there exists a Hilbert inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is continuously embedded in $(H, \langle ., . \rangle)$. That is, there exists an inner product $\langle ., . \rangle_*$ on M such that $(M, \langle ., . \rangle_*)$ is Hilbert and there exists $k > 0$ with

$$\langle x, x \rangle \leq k \, \langle x, x \rangle_* \quad \text{for all } x \in M.$$

Definition 17 ([5]).

An operator T on a Hilbert space $(H, \langle ., . \rangle)$ is said to be **semiclosed** *if its graph*

$$G(T) = \{(x, Tx) : x \in D(T)\}$$

is a semiclosed subspace of the Hilbert space $H \times H$.

Quotient and semiclosed operators are same.

Let $A \in B(H)$ and $B \in B(H)$ with $N(A) \subseteq N(B)$. Then $T = B/A$ is a linear operator, and the following are true:

1. T is semiclosed, that is, T is an algebraic combination of closed linear operators on H; every semiclosed operator is of the form $T = B/A$ and reciprocally.
2. T is closed if and only if the space if $R(A^*) + R(B^*)$ is closed.
3. T is bounded if and only if $R(A^*) \subseteq R(B^*)$.

The theory of semiclosed operators is so rich that much of the theory of closed operators is modeled after it. We have seen that semiclosed and quotient of operators are one and the same. In the next lecture, we shall discuss results of semiclosed operators.

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